NORMANHURST BOYS HIGH SCHOOL

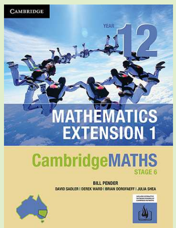
MATHEMATICS EXTENSION 1

YEAR 12 COURSE

Topic summary and exercises:

X1 X2 **Inverse Trigonometry**

With references to



Name: ....

Initial version by H. Lam, March 2014.
Last updated April 29, 2025, with major revision in March 2020 for Mathematics Extension 1.
Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under CC BY 2.0.

Symbols used

- Beware! Heed warning.
- Mathematics Advanced content.
- Mathematics Extension 1 content.
- Mathematics Extension 2 content.
- Literacy: note new word/phrase.
- Formulae/facts must be memorised.
- Available on the NESA Reference Sheet.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

ME11-3 applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems

ME12-4 uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution

Syllabus subtopics

ME-T1 Inverse Trigonometric Functions

ME-C2 Further Calculus Skills

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Advanced* or *CambridgeMATHS Year 12 Extension 1* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

✓ **Content/learning intentions**

☐ 24.1 Recognise increasing and decreasing functions.

☐ 24.2 Understand the difference between inverse relations and inverse functions.

☐ 24.3 Recognise the

- the inverse of an increasing or decreasing function is a function.
- the inverse of an increasing function is an increasing function.
- the inverse of an decreasing function is a decreasing function.

☐ 24.4 Understand how to generate inverse functions by restricting the domain of the original function.

☐ 24.5 Define and use the inverse trigonometric functions. (ACMSM119)

- Understand and use the notation $\arcsin x$ and $\sin^{-1} x$ for the inverse function of $\sin x$ when $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (and similarly for $\cos x$ and $\tan x$) and understand when each notation might be appropriate to avoid confusion with the reciprocal functions.
- Use the convention of restricting the domain of $\sin x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, so the inverse function exists. The inverse of this restricted sine function is defined by: $y = \sin^{-1} x, -1 \leq x \leq 1$, and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- Use the convention of restricting the domain of $\cos x$ to $0 \leq x \leq \pi$, so the inverse function exists. The inverse of this restricted cosine function is defined by: $y = \cos^{-1} x, -1 \leq x \leq 1$, and $0 \leq y \leq \pi$.
- Use the convention of restricting the domain of $\tan x$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, so the inverse function exists. The inverse of this restricted tangent function is defined by: $y = \tan^{-1} x, x$ is a real number , and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- Classify inverse trigonometric functions as odd, even or neither odd nor even.

✓ Content/learning intentions

- [ ] 24.6 Sketch graphs of the inverse trigonometric functions.

  - Graph transformations of the inverse trigonometric functions.

- [ ] 24.7 Use the relationships $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$, $\cos(\cos^{-1} x) = x$ and $\cos^{-1}(\cos x) = x$, $\tan(\tan^{-1} x) = x$ and $\tan^{-1}(\tan x) = x$, where appropriate, and state the values of x for which these relationships are valid.

- [ ] 24.8 Prove and use the properties:

  - $\sin^{-1}(-x) = -\sin^{-1} x$
  - $\tan^{-1}(-x) = -\tan^{-1} x$
  - $\cos^{-1}(-x) = \pi - \cos^{-1} x$
  - $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$

- [ ] 24.9 Solve problems involving inverse trigonometric functions in a variety of abstract and practical situations.

- [ ] 24.10 Find derivatives of inverse functions by using the relationship $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

- [ ] 24.11 (B) Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

  - Tangents and normals
  - Curve Sketching
  - Optimisation

✓ Content/learning intentions

☐ 24.12 Solve problems involving the derivatives of inverse trigonometric functions

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☐ 24.13 Integrate expressions of the form $\dfrac{1}{\sqrt{a^2 - x^2}}$ or $\dfrac{a}{a^2 + x^2}$ (ACMSM121)

☐ 24.14 (B) Calculate the area under a curve (ACMMM132)

☐ 24.15 (B) Calculate areas between curves determined by any functions within the scope of this syllabus (ACMMM134)

> Primitives which result in the inverse trigonometric functions in this case.

☐ 24.16 (B) Use the Trapezoidal rule to estimate areas under curves

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Section 1

Inverse Functions

Learning Goal(s)

Knowledge
Properties of inverse functions

Skills
Determining if a function has an inverse function

Understanding
The difference between inverse relations and inverse functions

By the end of this section am I able to:

24.1 Recognise increasing and decreasing functions.

24.2 Understand the difference between inverse relations and inverse functions.

24.3 Recognise the (properties of an increasing/decreasing function and its inverse equivalent)

24.4 Understand how to generate inverse functions by restricting the domain of the original function.

- Mutual inverse (over restricted domain)

$$x \to \boxed{f} \xrightarrow{f(x)} \boxed{f^{-1}} \to x \qquad x \to \boxed{f^{-1}} \xrightarrow{f^{-1}(x)} \boxed{f} \to x$$

- Domain/range swap:

- Reflection about $y = x$.
  - $f(x)$ and $f^{-1}(x)$ intersect on $y = x$
  - If f and f^{-1} intersect more than once, at least one intersection is on $y = x$.
    (Counterexample: $y = -x^3$)

- $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$

Theorem 1

Existence of an inverse function to $f(x)$ over a specified domain: $f(x)$ must be one to one within the specified domain. Methods of showing a function is one to one:

- Graphical, with a worded description, or
- Algebraically, show that $f(x)$ is
  - always increasing, i.e. $f'(x) > 0$, or
  - always decreasing, i.e. $f'(x) < 0$

Worded description:

Function is always increasing or always decreasing, therefore is one to one, therefore an inverse function exists over the specified domain.

Example 1

[2023 Ext 1 HSC Q14] Let $f(x) = 2x + \ln x$, for $x > 0$.

i. Explain why the inverse of $f(x)$ is a function. **1**

ii. Let $g(x) = f^{-1}(x)$. By considering the value of $f(1)$, or otherwise, evaluate $g'(2)$. **2**

Timed exam practice 1 (Allow approximately 1 minute)

[2024 CSSA Ext 1 Trial Q8] Given the function $f(x) = e^{-x} + ex$, $x \geq -1$.

What is the gradient of the inverse function $y = f^{-1}(x)$ at $(1, 0)$? **Answer:** (B)

(A) $e - 1$ (B) $\dfrac{1}{e-1}$ (C) $\dfrac{e^2-1}{e}$ (D) $\dfrac{e}{e^2-1}$

Example 2

[2021 Ext 1 HSC Q12] A function is defined by $f(x) = 4 - \left(1 - \frac{x}{2}\right)^2$ for x in the domain $(-\infty, 2]$.

i. Sketch the graph of $y = f(x)$ showing the x and y intercepts. **2**

ii. Find the equation of the inverse function, $f^{-1}(x)$, and state its domain. **3**

iii. Sketch the graph of $y = f^{-1}(x)$. **1**

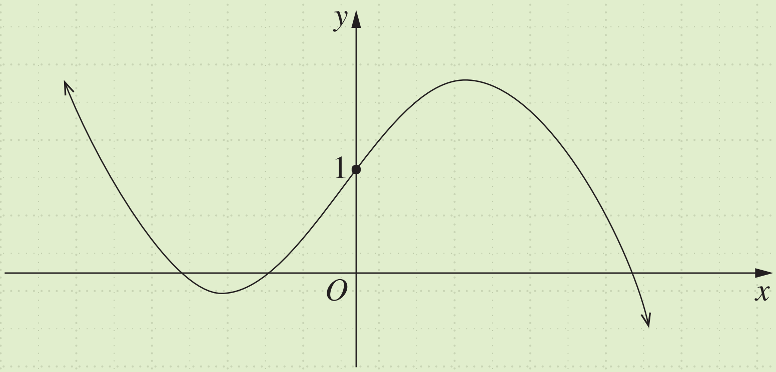
Answer: $y = 2 - 2\sqrt{4-x}$, $(-\infty, 4]$

Important note

Don't just randomly select *any* solution arising from a quadratic! *Always* give a reason why you're choosing one solution over another.

Example 3

[2023 Ext 1 HSC Q9] The graph of a cubic function, $y = f(x)$, is given below.

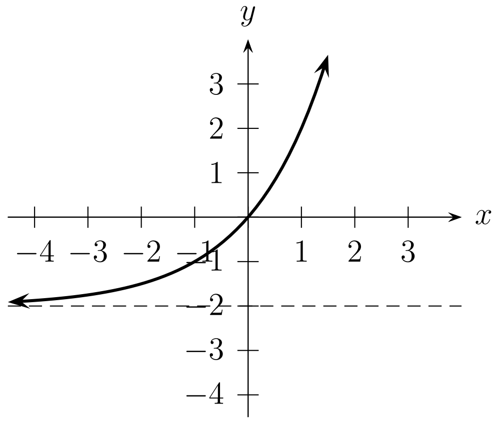


Which of the following functions has an inverse relation whose graph has more than 3 points with an x coordinate of 1? **Answer:** (D)

(A) $y = \sqrt{f(x)}$ (B) $y = \dfrac{1}{f(x)}$ (C) $y = f(|x|)$ (D) $y = |f(x)|$

1.0.1 Additional questions

1. **[2017 VCE Mathematical Methods Paper 2 Q4]** (16 marks)
Let $f : \mathbb{R} \mapsto \mathbb{R} : f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.

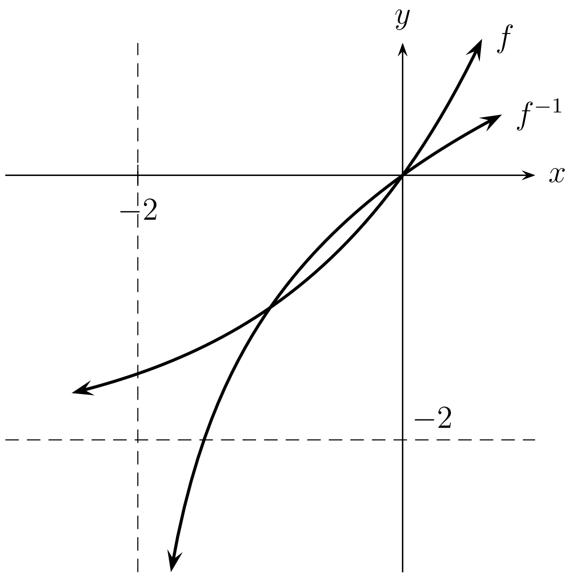


(a) (Modified to suit NSW syllabus) State the transformations required to the graph of $y = 2^x$ to obtain $y = 2^{x+1} - 2$. **2**

(b) Find the rule and domain for f^{-1}, the inverse function of f. **2**

(c) Show that $f(x)$ and $f^{-1}(x)$ intersect at $(-1, -1)$, and hence find the area bounded by the graphs of $f(x)$ and $f^{-1}(x)$. **3**

(d) Part of the graphs of f and f^{-1} are shown below. **2**



Find the gradient of f and the gradient of f^{-1} at $x = 0$.

The functions of g_k, where $k \in \mathbb{R}^+$, are defined with domain $\mathbb{R}$ such that $g_k = 2e^{kx} - 2$.

(e) Find the value of k such that $g_k(x) = f(x)$. **1**

(f) Find the rule for the inverse functions ${g_k}^{-1}$ of g_k, where $k \in \mathbb{R}^+$. **1**

(g) (Modified to suit NSW syllabus)

i. State the transformations required that transforms the graph of g_1 to the graph of g_k. **1**

ii. State the transformations required that transforms the graph of ${g_1}^{-1}$ to the graph of ${g_k}^{-1}$. **1**

(h) Omitted due to content not in NSW syllabus.

(i) ⚠ Let p be the value of k for which $g_k(x) = {g_k}^{-1}(x)$ has only one solution.

i. Find p. **2**

ii. Let $A(k)$ be the area bounded by the graphs of g_k and ${g_k}^{-1}$ for all $k > p$. **1**

State the smallest value of b such that $A(k) < b$.

Answers

(a) Translated 1 left and 2 down. (b) $f^{-1}(x) = \log_2(x+2) - 1 = \frac{1}{\ln 2}\ln(x+2)1$ (c) $\frac{-2}{\ln 2} + 3$ (d) $f'(0) = 2\ln 2$, $\left(f^{-1}\right)'(0) = \frac{1}{2\ln 2}$ (e) $k = \ln 2$ (f) ${g_k}^{-1}(x) = \frac{1}{k}\ln\left(\frac{x+2}{2}\right)$ (g) i. $g(x) = 2e^x - 2$, $g_k(x) = 2e^{kx} - 2$. Dilation of a factor of $\frac{1}{k}$ from the y axis. ii. Dilation of a factor of $\frac{1}{k}$ from the x axis. (h) Not in syllabus. (i) i. $p = \frac{1}{2}$ ii. $b = 4$

Further exercises

Ex 17A (Pender et al., 2019a)

- Q15-17, 18-20

Section 2

Inverse trigonometric functions

Learning Goal(s)

Knowledge
Properties of inverse trigonometric functions

Skills
Operate with the inverse trigonometric functions

Understanding
Properties of inverse trigonometric functions can be derived from the original trigonometric function

By the end of this section am I able to:

24.5 Define and use the inverse trigonometric functions

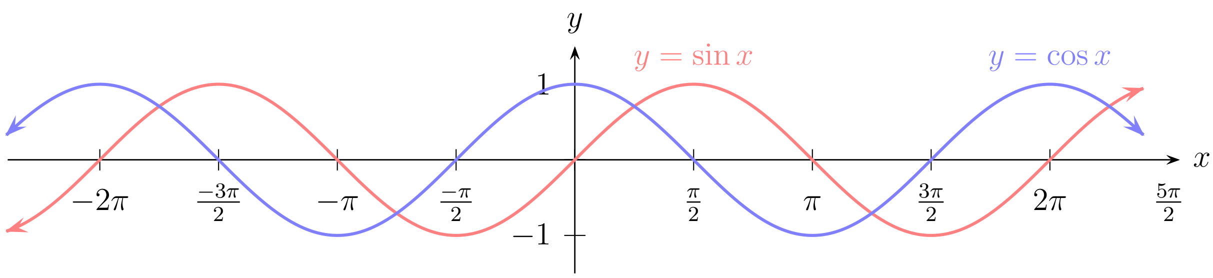
24.6 Sketch graphs of the inverse trigonometric functions.

24.7 Use the relationships $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$, $\cos\left(\cos^{-1} x\right) = x$ and $\cos^{-1}(\cos x) = x$, $\tan\left(\tan^{-1} x\right) = x$ and $\tan^{-1}(\tan x) = x$, where appropriate, and state the values of x for which these relationships are valid.

24.8 Prove and use the properties

24.9 Solve problems involving inverse trigonometric functions in a variety of abstract and practical situations.

2.1 Graphs of inverse trigonometric functions



$y = \sin x$ and $y = \cos x$ are ......many...... ....to.... ....one.... functions, thereby fails to have an inverse *function*.

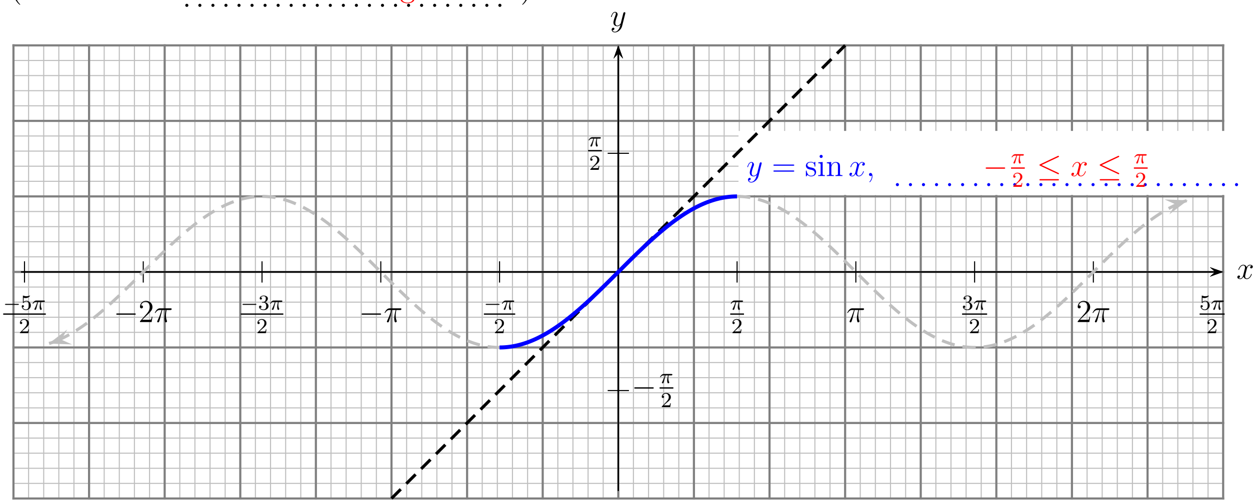
Solution

Fill in the spaces

- ......Restrict...... the ......domain...... of $y = \sin x$ and $y = \cos x$ so that they become ....one.... ....to.... ....one.... (or ......monotonic...... increasing/decreasing) to obtain an inverse.
- Want first quadrant values (in terms of π) included for simplicity.

2.1.1 Inverse sine

- Restrict $y = \sin x$ to between $x =$ ..$-\frac{\pi}{2}$.. to $x =$ ..$+\frac{\pi}{2}$..
  (monotonic ....increasing........)



- Domain/range:

  ➋ $D_f =$ .... ➋ $D_{f^{-1}} =$ ....

  ➋ $R_f =$ .... ➋ $R_{f^{-1}} =$ ....

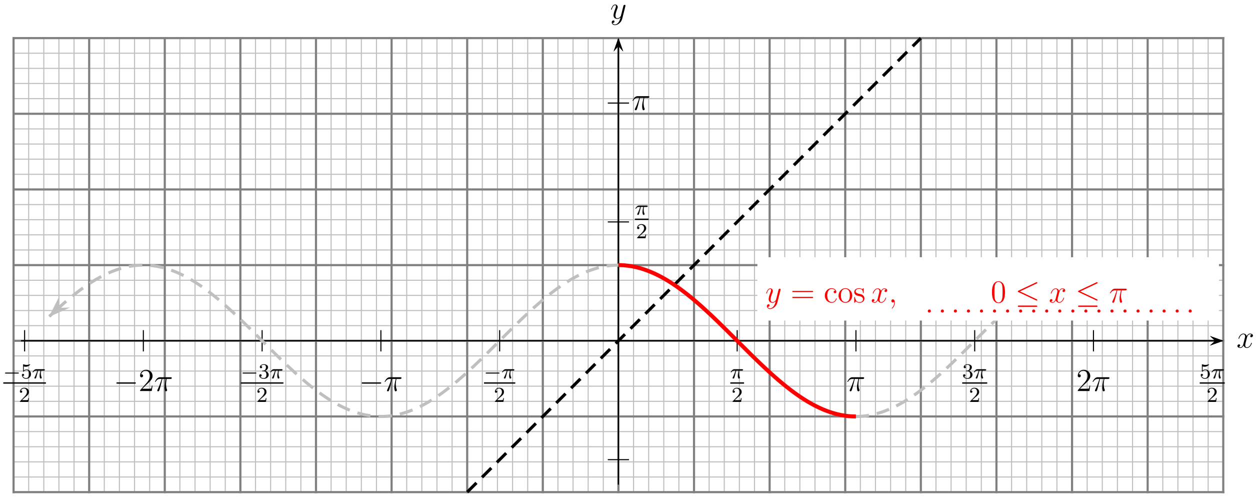
- Draw graph of inverse on grid

⚠ $y = \sin^{-1} x$ (or $y = \arcsin x$) is *not* the inverse relation to ....$y = \sin x$.... ; rather it is the inverse ....function.... to ....$y = \sin x$.... for ....$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.... .

- Draw general graph (diagrammatic) of $y = \sin^{-1} x$:

2.1.2 Inverse cosine

- Restrict $y = \cos x$ to between $x =$ 0 to $x =$ π (monotonic decreasing)



- Domain/range:

  - $D_f =$ .... $D_{f^{-1}} =$ ....

  - $R_f =$ .... $R_{f^{-1}} =$ ....

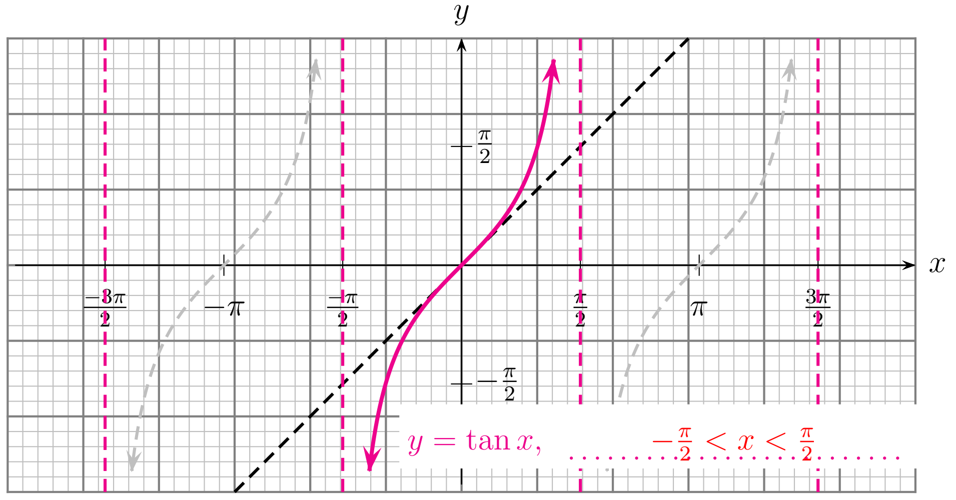
- Draw graph of inverse on grid.

⚠ $y = \cos^{-1} x$ (or $y = \arccos x$) is *not* the inverse relation to $y = \cos x$; rather it is the inverse function to $y = \cos x$ for $0 \le x \le \pi$.

- Draw general graph (diagrammatic) of $y = \cos^{-1} x$:

2.1.3 Inverse tan

- Restrict $y = \tan x$ to between $x =$ $-\frac{\pi}{2}$ to $x =$ $\frac{\pi}{2}$
  (monotonic increasing)



- Domain/range:

  ➋ $D_f =$ .... ➋ $D_{f^{-1}} =$ ....

  ➋ $R_f =$ .... ➋ $R_{f^{-1}} =$ ....

- Draw graph of inverse on grid.

⚠ $y = \tan^{-1} x$ (or $y = \arctan x$) is *not* the inverse relation to $y = \tan x$; rather it is the inverse function to $y = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

- Draw general graph (diagrammatic) of $y = \tan^{-1} x$:

Example 4

[2022 Ext 1 HSC Q13] The function f is defined by $f(x) = \sin x$ for all real numbers x. Let g be the function defined on $[-1, 1]$ by $g(x) = \arcsin x$.

Is g the inverse of f? Justify your answer.

2.1.4 Curve sketching

Example 5

Sketch the following curves:

1. $y = \sin^{-1} 2x$
2. $y = 2\cos^{-1} x$
3. $y = \frac{1}{2}\tan^{-1} \frac{x}{3}$
4. $y = \frac{1}{4}\cos^{-1}(2x+1)$
5. ⚠ $y = 2\cos^{-1}(3x-2)+1$

Important note

Tip Change the axes and scale as needed, instead of changing the shape of the graph.

Timed exam practice 2 (Allow approximately 1 minute)

[2024 Ext 1 HSC Q5] Consider the function $g(x) = 2\sin^{-1}(3x)$.

Which transformations have been applied to $f(x) = \sin^{-1} x$ to obtain $g(x)$?

(A) Vertical dilation by factor of $\dfrac{1}{2}$ and a horizontal dilation of $\dfrac{1}{3}$

(B) Vertical dilation by factor of $\dfrac{1}{2}$ and a horizontal dilation of 3

(C) Vertical dilation by factor of 2 and a horizontal dilation of $\dfrac{1}{3}$

(D) Vertical dilation by factor of 2 and a horizontal dilation of 3

Example 6

[2010 CSSA Ext 1] (2 marks) By considering the graph of $y = \sin^{-1} x$ or otherwise, show that the equation $\sin^{-1} x + x - \frac{\pi}{2} = 0$ has only one real and positive root.

2.1.5 Additional questions

1. Sketch the following graphs:

(a) $y = 3\sin^{-1} 3x - 1$ (b) $y = 3\tan^{-1} \frac{x}{2}$

2. [**2021 CSSA Ext 1 Trial Q5**] Which of the following function has a domain $[1, 5]$ and range $[1, 4\pi + 1]$?

(A) $f(x) = 2\sin^{-1}\left(\dfrac{x-3}{2}\right) + 1$ (C) $f(x) = 4\sin^{-1}\left(\dfrac{x-3}{2}\right) + 1$

(B) $f(x) = 4\cos^{-1}\left(\dfrac{x-3}{2}\right) + 1$ (D) $f(x) = 2\cos^{-1}\left(\dfrac{x-3}{2}\right) + 1$

3. [**2022 CSSA Ext 1 Trial Q7**] Which of the following is an odd function?

(A) $a(x) = \dfrac{\pi}{2} + \cos^{-1} x$ (C) $c(x) = \dfrac{\pi}{2} + \cos^{-1}(1 - x)$

(B) $b(x) = \dfrac{\pi}{2} - \cos^{-1} x$ (D) $d(x) = \dfrac{\pi}{2} - \cos^{-1}(1 - x)$

4. [**2020 Ext 1 HSC Sample Q10**] The graph of the function $y = \sin^{-1}(x - 4)$ is transformed by being dilated horizontally with a scale factor of 2 and then translated to the right by 1.

What is the equation of the transformed graph?

(A) $y = \sin^{-1}\left(\dfrac{x-9}{2}\right)$ (C) $y = \sin^{-1}(2x - 6)$

(B) $y = \sin^{-1}\left(\dfrac{x-10}{2}\right)$ (D) $y = \sin^{-1}(2x - 5)$

5. [**2021 Independent Ext 1 Trial Q13**]

i. Find the domain and range of the function $y = \cos^{-1}\left(\dfrac{x-1}{2}\right) - \dfrac{\pi}{2}$ **2**

ii. Sketch the graph of the function $y = \cos^{-1}\left(\dfrac{x-1}{2}\right) - \dfrac{\pi}{2}$ showing clearly the coordinates of any endpoints and intercepts on the coordinate axes. **2**

iii. Describe a transformation of the graph of the function $y = \cos^{-1}\left(\dfrac{x-1}{2}\right) - \dfrac{\pi}{2}$ that would change it into the graph of an odd function. **1**

6. **[2022 Ext 1 HSC Q9]** A given function $f(x)$ has an inverse $f^{-1}(x)$.

The derivatives of $f(x)$ and $f^{-1}(x)$ exist for all real numbers x.

The graphs $y = f(x)$ and $y = f^{-1}(x)$ have at least one point of intersection.

Which statement is true for all points of intersection of these graphs?

(A) All points of intersection lie on the line $y = x$.

(B) None of the points of intersection lie on the line $y = x$.

(C) At no point of intersection are the tangents to the graphs parallel.

(D) At no point of intersection are the tangents to the graphs perpendicular.

Answers

1. Check with online graphing tools. **2.** (B) **3.** (B) **4.** (A) **5.** i. $D = [-1, 3]$, $R = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ii. Check with online graphing tools iii. Translate horizontally to the left by 1 unit **6.** (D)

Further exercises

Ex 17C (Pender et al., 2019a)

- Q1, 2, 3, 6

Further exercises (Legacy Textbooks)

Ex 1C

- Q1-15

2.1.6 Harder graphs

Example 7

Sketch $y = \sin\left(\sin^{-1} x\right)$.

Steps

1. Write down domain/range of inner & outer functions:

2. Draw chain diagram, noting range of ....inner.... function becomes domain of ....outer.... function:

3. Sketch graph:

Example 8

Sketch $y = \cos\left(\cos^{-1} x\right)$.

Example 9

Sketch $y = \tan\left(\tan^{-1} x\right)$.

Example 10

Sketch $y = \sin^{-1}\left(\sin x\right)$

Example 11

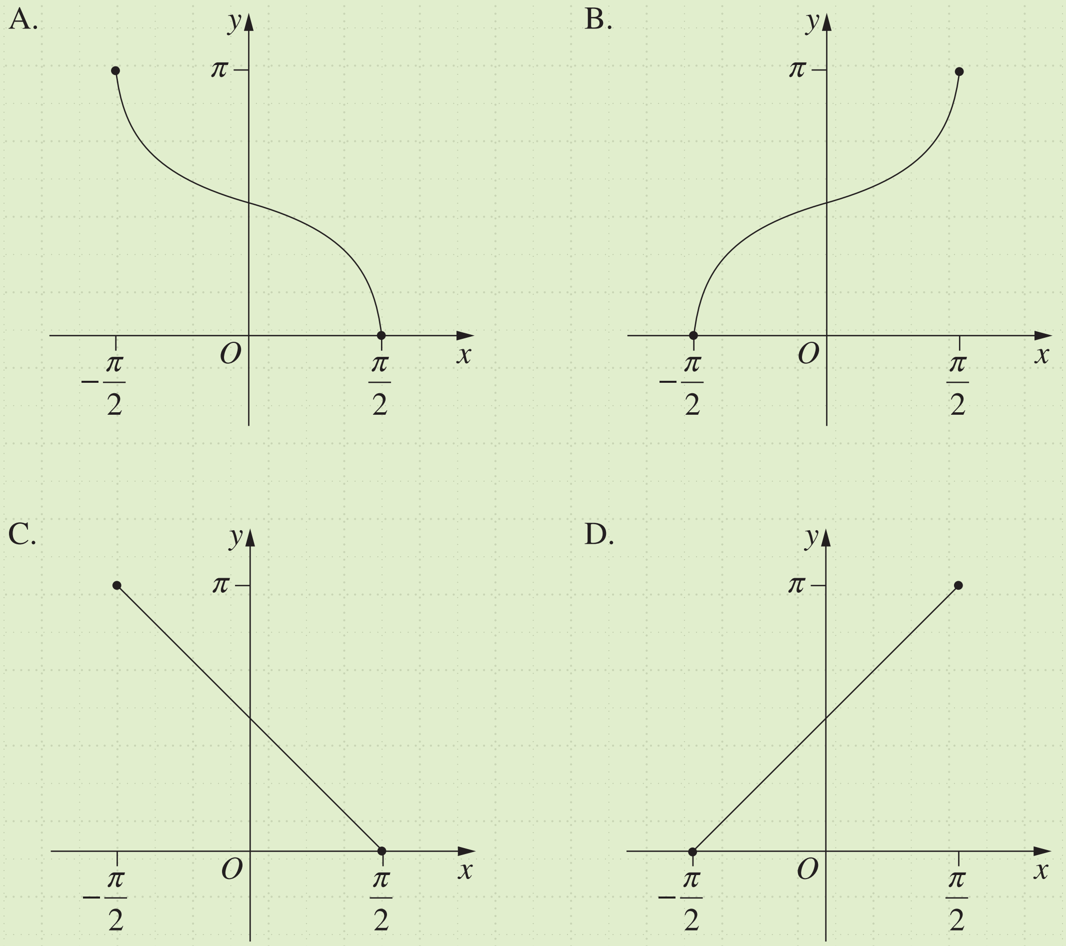
Sketch $y = \cos^{-1}(\cos x)$

Example 12

Sketch $y = \cos(\sin^{-1} x)$.

Example 13

[2019 Ext 1 HSC Q9] Which graph best represents $y = \cos^{-1}(-\sin x)$, for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$?



Further exercises

Ex 17C (Pender et al., 2019a)

- Q7-11

Further exercises (Legacy Textbooks)

Ex 1C

- Q11-15

2.2 Properties of inverse trigonometric functions

Laws/Results

Inverse trigonometric functions produce ....angles........ in the following quadrants:

Function	Quadrants
$\sin^{-1} x$	....First.... and '....negative.... ....first....'
$\cos^{-1} x$	....First.... and ....second....
$\tan^{-1} x$	....First.... and '....negative.... ....first....'

- '....Negative.... ....first....' quadrant: $y \in \left[-\frac{\pi}{2}, 0\right]$
- ....First.... quadrant: $y \in \left[0, \frac{\pi}{2}\right]$
- ....Second.... quadrant: $y \in \left[\frac{\pi}{2}, \pi\right]$

Check the range of the graphs of the inverse trigonometric functions on pages 15, 16 and 17 respectively.

2.2.1 Exact value problems

Example 14

Evaluate, writing as an exact value:

1. $\sin^{-1} \frac{\sqrt{3}}{2}$.

2. $\cos^{-1} \left(-\frac{\sqrt{3}}{2}\right)$.

3. $\tan^{-1} \left(-\frac{1}{\sqrt{3}}\right)$

4. $\sin^{-1} \left(-\frac{1}{\sqrt{2}}\right)$

5. $\cos^{-1} \left(-\frac{1}{\sqrt{2}}\right)$

Example 15

Evaluate, writing as an exact value:

1. $\tan\left(\sin^{-1}\frac{1}{2}\right)$

2. $\cos\left(\tan^{-1}1\right)$

3. $\cos^{-1}\left(\sin\frac{\pi}{3}\right)$

4. $\tan^{-1}\left(\sqrt{6}\sin\frac{\pi}{4}\right)$

5. $\sin^{-1}\left(\cos\frac{7\pi}{4}\right)$

6. $\cos^{-1}\left(\sin\left(-\frac{2\pi}{3}\right)\right)$

Example 16

Evaluate, writing as an exact value:

1. $\sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$

2. $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$

3. $\cos^{-1}\left(\cos\frac{5\pi}{6}\right)$

4. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

5. $\tan^{-1}\left(\tan\frac{7\pi}{4}\right)$

6. $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$

Important note

Draw a chain diagram if necessary.

Example 17

[2007 Ext 1 HSC Q5] Find the exact values of x and y which satisfy the simultaneous equations

$$\begin{cases} \sin^{-1} x + \dfrac{1}{2}\cos^{-1} y = \dfrac{\pi}{3} \\ 3\sin^{-1} x - \dfrac{1}{2}\cos^{-1} y = \dfrac{2\pi}{3} \end{cases}$$

Answer: $x = \frac{1}{\sqrt{2}}$, $y = \frac{\sqrt{3}}{2}$

2.2.2 Triangle problems

Important note

Draw .... picture .... !

Example 18

Evaluate $\sin\left(\cos^{-1}\frac{1}{5}\right)$, writing as an exact value.

Example 19

Evaluate $\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{12}{5}\right)\right]$, writing as an exact value. **Answer:** $-\frac{33}{65}$

Example 20

Find the exact value of $\cos\left(2\sin^{-1}\left(-\frac{3}{5}\right)\right)$. **Answer:** $\frac{7}{25}$

Example 21

Prove $2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4}$.

Example 22

Find the exact value of $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5}$. **Answer:** $\frac{\pi}{4}$

Example 23

Prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

Further exercises

Ex 17B (Pender et al., 2019a)

- Q2-7, 14

Further exercises (Legacy Textbooks)

Ex 1B (Pender et al., 2000)

- Q2, 4-23

2.2.3 Identities

Example 24

Show that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

Example 25

[2010 CSSA Ext 1] Find an expression for y in terms of x if for $x > 0$ and $y > 0$,

$$\tan^{-1} x = \tan^{-1} y + \frac{\pi}{4}$$

Answer: $y = \frac{x-1}{x+1}$

Example 26

Prove $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$.

Timed exam practice 3 (Allow approximately 5 minutes)

[2024 Ext 1 HSC Q14] (3 marks)

i. Explain why the equation $\tan^{-1}(3x) + \tan^{-1}(10x) = \theta$, where $-\pi < \theta < \pi$, has exactly one solution. **1**

ii. Solve $\tan^{-1}(3x) + \tan^{-1}(10x) = \dfrac{3\pi}{4}$. **2**

Marking criteria

i. ✓ [1] Provides correct explanation

ii. ✓ [1] Obtains correct quadratic equation for x, or equivalent merit

✓ [2] Provides correct solution

2.2.4 Additional questions

1. **[2023 Ext 1 HSC Q5]** Which of the following is the value of $\sin^{-1}(\sin a)$ given that $\pi < a < \dfrac{3\pi}{2}$?

(A) $a - \pi$ (B) $\pi - a$ (C) a (D) $-a$

2. **[2023 Ext 1 HSC Q7]** Which statement is always true for real numbers a and b where $-1 \le a \le b \le 1$?

(A) $\sec a < \sec b$

(B) $\sin^{-1} a < \sin^{-1} b$

(C) $\arccos a < \arccos b$

(D) $\cos^{-1} a + \sin^{-1} a < \cos^{-1} b + \sin^{-1} b$

3. **[2009 Independent Ext 1 Trial]** If $\cos^{-1} x - \sin^{-1} x = k$, where $-\frac{\pi}{2} \le k \le \frac{3\pi}{2}$, show that $x = \frac{1}{\sqrt{2}}\left(\cos\frac{k}{2} - \sin\frac{k}{2}\right)$.

4. **[2011 HGHS Ext 1 Trial]** It is known that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute angles.

(i) Show that $\sin\left(\sin^{-1} x - \cos^{-1} x\right) = 2x^2 - 1$. **2**

(ii) Hence or otherwise, solve the equation **2**

$$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$$

giving your solution(s) as exact values.

Answers

1. (B) **2.** (B) **3.** Show **4.** $\frac{\sqrt{17}-1}{4}$

Further exercises

Ex 17B (Pender et al., 2019b)

- Q8-13

Section 3

Differentiation of Inverse Trigonometric Functions

Learning Goal(s)

Knowledge
How to differentiate the inverse trigonometric functions

Skills
Using the chain rule to differentiate

Understanding
When to apply differentiation techniques from Mathematics Advanced

By the end of this section am I able to:

24.10 Find derivatives of inverse functions by using the relationship $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

24.11 Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

24.12 Solve problems involving the derivatives of inverse trigonometric functions

3.1 Derivatives of $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$

Laws/Results

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \tag{12.1}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \tag{12.2}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \tag{12.3}$$

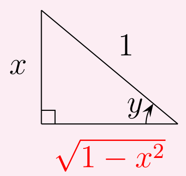
Steps

Proof of $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$**:**

1. Let $y = \sin^{-1} x$. Change subject to x:

$$x = \sin y \qquad -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}$$

2. Draw right angled triangle to represent situation:



3. Find $\dfrac{dx}{dy}$:

4. Use $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$:

Example 27

Following the same steps, give a proof of why $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$.

Example 28

Differentiate:

1. $y = 4\sin^{-1} 2x$ **2.** $y = \cos^{-1} \frac{x}{3}$ **3.** $y = 3\tan^{-1} \frac{x}{2}$

Example 29

Differentiate with respect to x:

1.	$x^2 \sin^{-1} x$	**3.**	$(\ln x + \cos^{-1} 2x)^4$	**5.**	$\cos^{-1}\left(2x\sqrt{1-x^2}\right)$
2.	$\sin^{-1}(e^{3x})$	**4.**	$\tan(\cos^{-1}\sqrt{x})$	**6.**	$\sqrt{\tan^{-1}(x^3)}$

Answers: **1.** $\frac{x^2}{\sqrt{1-x^2}} + 2x\sin^{-1} x$ **2.** $\frac{3e^{3x}}{\sqrt{1-e^{6x}}}$ **3.** $4\left(\ln x + \cos^{-1} 2x\right)^3\left(\frac{1}{x} - \frac{2}{\sqrt{1-4x^2}}\right)$. **4.** $-\frac{1}{2x\sqrt{x(1-x)}}$ **5.** $-\frac{2}{\sqrt{1-x^2}}$ **6.** $\frac{3x^2}{2(1+x^6)\sqrt{\tan^{-1} x^3}}$

Timed exam practice 4 (Allow approximately 5 minutes)

[2024 Ext 1 HSC Q14] (3 marks) For what values of the constant k would the function $f(x) = \dfrac{kx}{x^2+1} + \arctan x$ have an inverse?

Answer: $k \in [-1, 1]$

Marking criteria

✓ [1] Recognises that $f(x)$ needs to be monotonically increasing or monotonically decreasing, or equivalent merit

✓ [2] Correctly completes one of two cases, or equivalent merit

✓ [3] Provides correct solution

3.2 Applications of differentation

3.2.1 Equation of tangent/normal

Example 30

Find the gradient of the tangent to $y = \sin^{-1} 2x$ where $x = 0$.

Example 31

Find the equation of the normal to the curve $y = x \tan^{-1} x$ at $\left(1, \frac{\pi}{4}\right)$.

Answer: $16x + 4(2+\pi)y - 2\pi - \pi^2 - 16 = 0$

3.2.2 Curve sketching

Example 32

[2010 Ext 1 HSC] Let $f(x) = \tan^{-1} x + \tan^{-1}\left(\dfrac{1}{x}\right)$, $x \neq 0$.

(i) By differentiating $f(x)$ or otherwise, show that $f(x) = \dfrac{\pi}{2}$ for $x > 0$. **3**

(ii) Given that $f(x)$ is an odd function, sketch the graph of $f(x)$. **1**

Example 33

[2011 NSGHS Ext 1 Trial Q6]

i. Determine the gradient of the tangent to the curve $y = \tan^{-1}(e^x)$ at $x = 0$. **2**

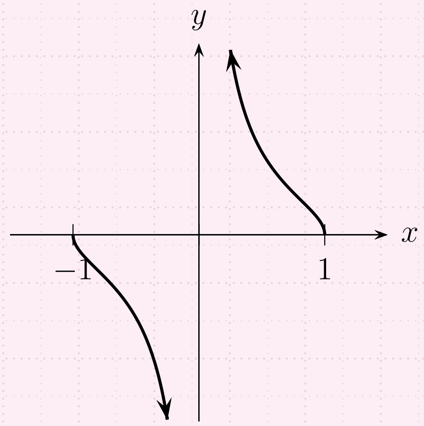
ii. Hence find the equation of the tangent at that point. **2**

iii. Discuss the behaviour of the curve as $x \to \infty$. **1**

Timed exam practice 5 (Allow approximately 11 minutes)

[2020 Ext 1 HSC Q13] Suppose $f(x) = \tan\left(\cos^{-1}(x)\right)$ and $g(x) = \dfrac{\sqrt{1-x^2}}{x}$.

The graph of $y = g(x)$ is given.



i. Show that $f'(x) = g'(x)$. **4**

ii. Using part (i) or otherwise, show that $f(x) = g(x)$. **3**

Marking criteria

i. ✓ [1] Attempts one derivative, or equivalent merit
✓ [2] Obtains one correct derivative or attempts both, or equivalent merit
✓ [3] Obtains one correct derivative and makes some progress towards obtaining the other, or equivalent merit
✓ [4] Provides correct solution

ii. ✓ [1] Observes that $f(x) - g(x) = 0$, or equivalent merit
✓ [2] Provides a correct solution for one part of the domain, or equivalent merit
✓ [3] Provides correct solution for both parts of the domain

3.2.3 Additional questions

1. Find the equation of the tangent to $y = \left(\sin^{-1} x\right)^3$ where $x = \frac{1}{2}$.

2. **[2002 Ext 1 HSC Q5]** Consider the function $f(x) = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$ for $0 \le x \le 1$.

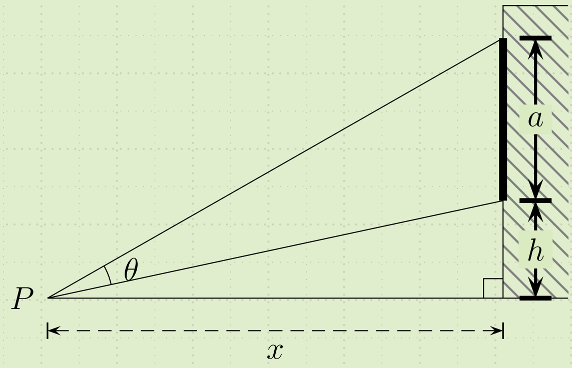
i. Show that $f'(x) = 0$ for $0 < x < 1$. **3**

ii. Sketch the graph of $y = f(x)$. **2**

3.2.4 Optimisation

Example 34

[2009 Ext 1 HSC Q7] A billboard of height a metres is mounted on the side of a building, with its bottom edge h metres above street level. The billboard subtends an angle θ at the point P, x metres from the building.



(i) Use the identity $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ to show that **2**

$$\theta = \tan^{-1}\left(\frac{ax}{x^2 + h(a+h)}\right)$$

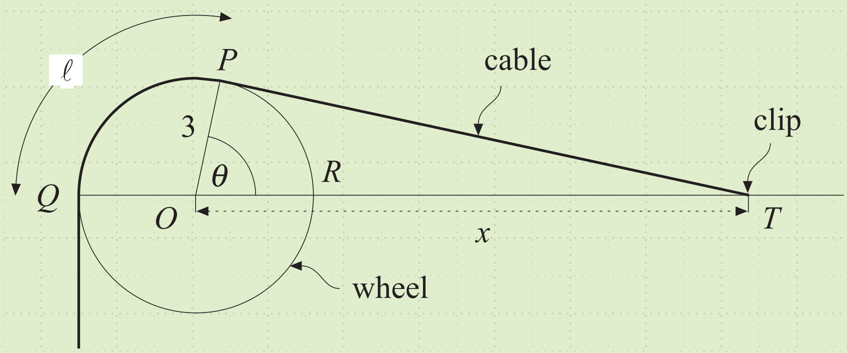
(ii) The maximum value of θ occurs when $\dfrac{d\theta}{dx} = 0$ and x is positive. **3**

Find the value of x for which θ is maximum.

3.2.5 Rates of change

Example 35

[1995 3U HSC Q6] A long cable is wrapped over a wheel of radius 3 metres and one end is attached to a clip at T. The centre of the wheel is at O, and QR is a diameter. The point T lies on the line OR at a distance x metres from O.



The cable is tangential to the wheel at P and Q as shown. Let $\angle POR = \theta$ (in radians).

The length of cable in contact with the wheel is ℓ metres; that is, the length of the arc between P and Q is ℓ metres.

(i) Explain why $\cos\theta = \dfrac{3}{x}$. **1**

(ii) Show that $\ell = 3\left[\pi - \cos^{-1}\left(\dfrac{3}{x}\right)\right]$. **2**

(iii) Show that $\dfrac{d\ell}{dx} = -\dfrac{9}{x\sqrt{x^2-9}}$. **2**

What is the significance of the fact that $\dfrac{d\ell}{dx}$ is negative?

(iv) Let $s = \ell + PT$. **2**

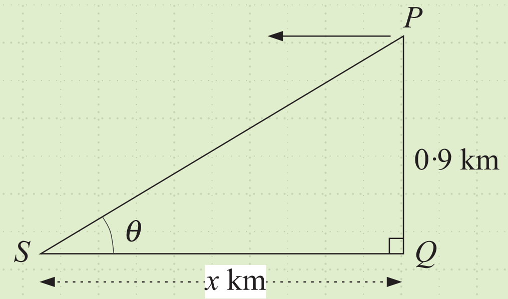
Express s in terms of x.

(v) The clip at T is moved away from O along the line OR at a constant speed of 2 metres per second. **2**

Find the rate at which s changes when $x = 10$. **Answer:** $\frac{\sqrt{91}}{5}$ ms^{-1}

Example 36

[1997 3U HSC Q4] A searchlight on the ground at S detects and tracks a plane P that is due east of the searchlight. The plane is flying due west at a constant velocity of 240 kilometres per hour and maintains a constant height of 900 metres above ground level.



Let $\theta(t)$ be the angle of elevation of the plane at time t seconds and let $x(t)$ kilometres be the distance of S to the point Q on the ground directly below P.

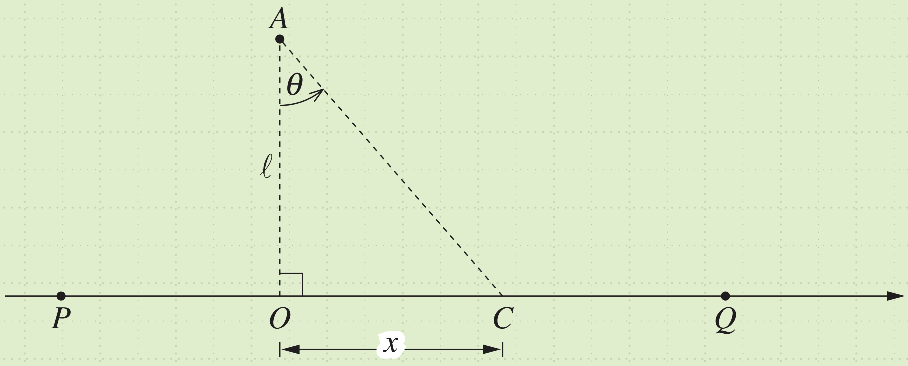
(i) Show that $\dfrac{dx}{d\theta} = -\dfrac{0.9}{\sin^2\theta}$ **2**

(ii) Show that the rate of change of the angle of elevation of the plane when $\theta = \dfrac{\pi}{4}$ is equal to $\dfrac{1}{27}$ radians per second. **2**

3.2.6 Motion

Example 37

[2008 Ext 1 HSC Q3] A race car is travelling on the x axis from P to Q at a constant velocity v.



A spectator is at A which is directly opposite O, and $OA = \ell$ metres. When the car is at C, its displacement from O is x metres and $\angle OAC = \theta$, with $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

i. Show that $\dfrac{d\theta}{dt} = \dfrac{v\ell}{\ell^2 + x^2}$. **2**

ii. Let m be the maximum value of $\dfrac{d\theta}{dt}$. **1**

Find the value of m in terms of v and ℓ.

iii. There are two values of θ for which $\dfrac{d\theta}{dt} = \dfrac{m}{4}$. **2**

Find these two values of θ.

Further exercises

Ex 12A (Pender et al., 2019b)

- All questions

Further exercises (Legacy Textbooks)

Ex 1D

- Q2 last 2 columns
- Q3-15
- X2 Q16-20

Section 4

Integration resulting in Inverse Trigonometric Functions

Learning Goal(s)

Knowledge
How to integrate expressions which result in the inverse trigonometric functions

Skills
Using various integration techniques

Understanding
When to apply various integration techniques from Mathematics Advanced

By the end of this section am I able to:

24.13 Integrate expressions of the form $\dfrac{1}{\sqrt{a^2-x^2}}$ or $\dfrac{a}{a^2+x^2}$

24.14 Calculate the area under a curve

24.15 Calculate areas between curves determined by any functions within the scope of this syllabus

24.16 Use the Trapezoidal rule to estimate areas under curves

4.1 Simple integrals

Laws/Results

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\tfrac{x}{a}\right)+C \tag{16.1}$$

$$\int -\frac{1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\tfrac{x}{a}\right)+C \tag{16.2}$$

$$\int \frac{1}{a^2+x^2}\,dx = \tfrac{1}{a}\tan^{-1}\left(\tfrac{x}{a}\right)+C \tag{16.3}$$

Example 38

Evaluate:

(a) $\displaystyle\int \frac{2}{\sqrt{9-x^2}}\,dx.$

(b) $\displaystyle\int \frac{6}{49+25x^2}\,dx.$

(c) $\displaystyle\int \frac{1}{8+x^2}\,dx.$

(d) $\displaystyle\int \frac{1}{\sqrt{5-3x^2}}\,dx.$

Example 39

Evaluate:

(a) $\displaystyle\int_0^2 \frac{4}{\sqrt{4-x^2}}\,dx.$

(b) $\displaystyle\int_0^{\frac{3}{2}} \frac{4}{\sqrt{9-2x^2}}\,dx.$

4.2 Harder integrals

Example 40

[**JRAHS 1993 3U Trial**] Evaluate $\displaystyle\int \frac{x^2+1}{x^2+4}\,dx$.

Example 41

[**2010 NSB Ext 1 Trial**]

(i) Differentiate $\dfrac{2x}{4+x^2} + \tan^{-1}\left(\dfrac{x}{2}\right)$. **2**

(ii) Hence evaluate $\displaystyle\int_0^2 \frac{dx}{\left(4+x^2\right)^2}$. **2**

Example 42

(a) Differentiate $x \sin^{-1} x$.

(b) Hence, find a primitive for $\sin^{-1} x$.

Example 43

[1993 Abbotsleigh 3U Trial] Find $\dfrac{d}{dx}(x \tan^{-1} x)$ and hence show that

$$\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$$

Quadratics in denominators

- Transform quadratics in denominators into integrands that resemble derivative of $\sin^{-1}$ and $\tan^{-1}$.

Example 44

Evaluate $\displaystyle\int \frac{dx}{x^2+2x+2}$.

Answer: $\tan^{-1}(x+1)+C$

Example 45

Evaluate $\displaystyle\int \frac{3}{\sqrt{6x-x^2}}\,dx$.

4.3 Applications of integration

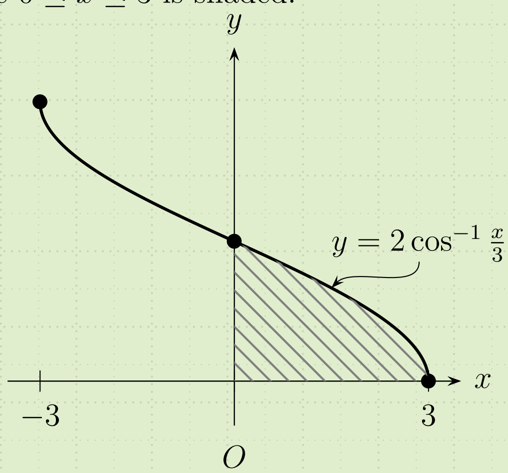
4.3.1 Area problems

Example 46

Find the exact area of the region bounded by the curve $y = \dfrac{1}{1+x^2}$, the coordinate axes, and the line $x = 3$.

Example 47

[2001 Ext 1 HSC Q5] The sketch shows the curve $y = f(x)$ where $f(x) = 2\cos^{-1}\frac{x}{3}$. The area under the curve $0 \le x \le 3$ is shaded.



i. Find the y intercept. **1**

ii. Determine the inverse function $y = f^{-1}(x)$, and write down the domain D of this inverse function. **2**

iii. Calculate the area of the shaded region. **2**

Example 48

Find the area bounded by the curve $y = 3\sin^{-1}\left(\frac{x}{4}\right)$, the x axis and the line $x = 4$.

Important note

- The primitives of $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$ are *not* a part of the course.
- Use y axis to assist where necessary.

Example 49

[1993 JRAHS 3U Trial]

(i) Neatly sketch the graph of $y = \sin^{-1} x$, stating its domain and range.

(ii) By considering the graph in (i) or otherwise, find the exact value of

$$\int_0^{\frac{1}{2}} \sin^{-1} x \, dx$$

Example 50

[2019 Fort Street HS Trial HSC Q14]

i. Find $\dfrac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$. **2**

ii. Hence, find the area between the curve $y = \cos x$, the y axis and the lines $y = \frac{1}{2}$ and $y = 1$. **2**

Answer: $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$

4.3.2 Motion/Rates of change

Example 51

[Ex 9F Q9] As a particle moves around a circle, its angular velocity is given by

$$\frac{d\theta}{dt} = \frac{1}{1+t^2}$$

(a) Given that the particle starts at $\theta = \dfrac{\pi}{4}$, find θ as a function of t.

(b) Hence find t as a function of θ.

(c) Using the result from part (a), show that $\dfrac{\pi}{4} \leq \theta < \dfrac{3\pi}{4}$, and hence explain why the particle never moves through an angle of more than $\dfrac{\pi}{2}$.

Further exercises

Ex 12B

- Q1-20

Further exercises (Legacy Textbooks)

Ex 1E

- Q2, 3 last 2 columns
- Q4-24

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

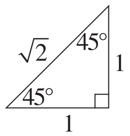
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

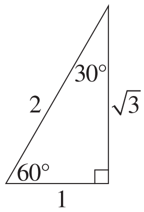
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$





Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

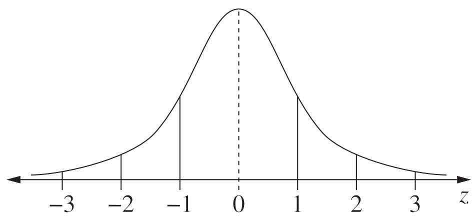
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x-\mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X-\mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1-p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1-p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^{n} &= r^{n}(\cos n\theta + i\sin n\theta) \\ &= r^{n}e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^{2}(x - c)$$

References

Fitzpatrick, J. B., & Aus, B. (2019). *New Senior Mathematics – Extension 1 for Years 11 & 12*. Pearson Education.

Pender, W., Sadler, D., Shea, J., & Ward, D. (2000). *Cambridge Mathematics 3 Unit Year 12* (1st ed.). Cambridge University Press.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019a). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 11* (1st ed.). Cambridge Education.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019b). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 12* (1st ed.). Cambridge Education.